Learning theory in the "physical" corner of the Hilbert space

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October 31, 2024

Upcoming section

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(Alexander Pope, ChatGPT)

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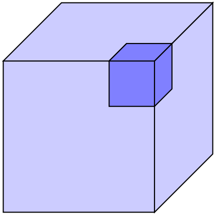
- However, physical systems live in a much smaller "corner", where the locality of natural laws prevails.
- This has a major impact on our understanding of the 'learnability' of physical systems.



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- A collection of quantum states on n qubits, $\mathcal{C}_n$.

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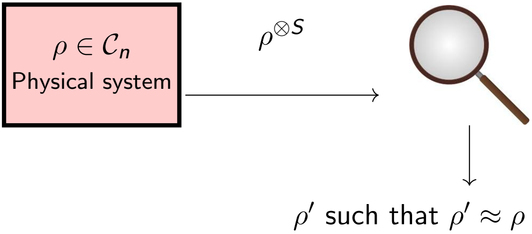

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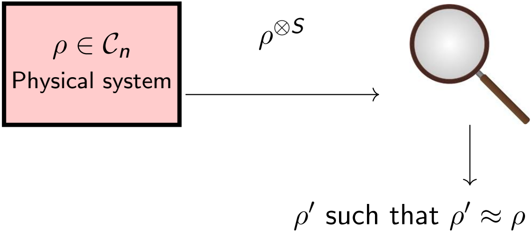

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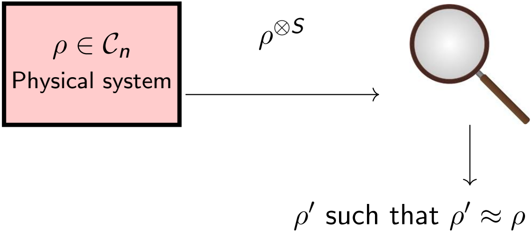

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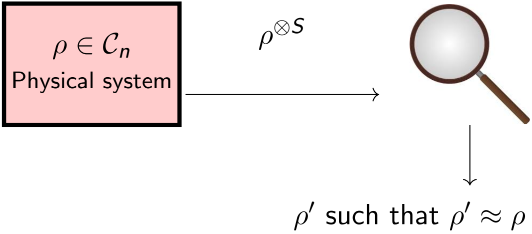

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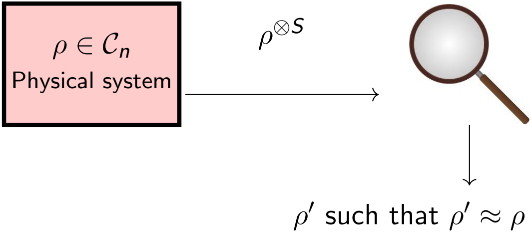

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- Time-efficient quantum measurements?
- Quantum measurements that only correlate a few qubits?

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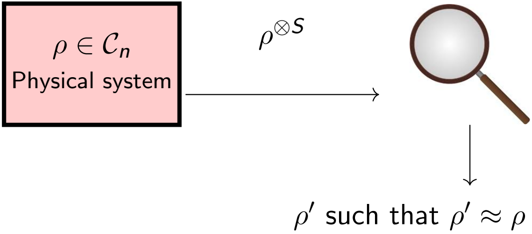

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Physical quantum states

- When $\mathcal{C}_n =$ the set of all quantum states, each quantum state is specified by $\approx 2^{2n}$ parameters. The sample complexity of learning an unknown general quantum state scales as $\approx 2^{2n}$ ([O'Donnell, Wright 2016], [Haah, Harrow, Ji, Wu, Yu, 2016]).

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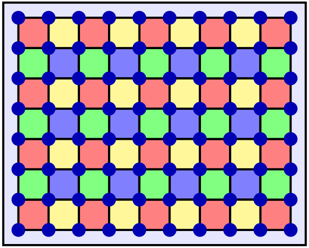
- In physical settings, the number of relevant parameters turn out to be $\mathrm{poly}(n)$. Two sources:
  - Experiments can only probe $\mathrm{poly}(n)$ parameters.
  - Locality in physical systems.

Quantum many-body systems



- A fundamental way locality arises in quantum many-body systems is via interaction between qubits, described by a local Hamiltonian $H(\mu) = \sum_i \mu_i E_i$ (where, E_i are a basis of operators, such as Pauli operators).

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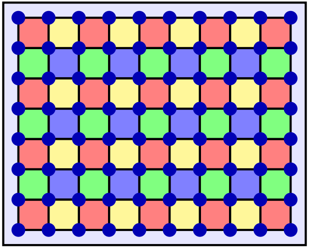

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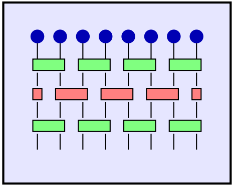
- The local nature of interactions also forces that elementary quantum gates act locally.

Widely studied quantum many-body systems

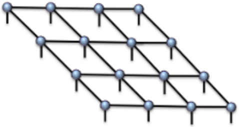
Gibbs state (specified by the Hamiltonian's coefficients μ)

$$\rho_\beta(\mu) = \frac{e^{-\beta H(\mu)}}{Z_\beta(\mu)}.$$

A quantum circuit of depth t (specified by $O(nt)$ local unitaries)



A tensor network state (specified by local tensors of dimension $d \times D \times D \dots$)



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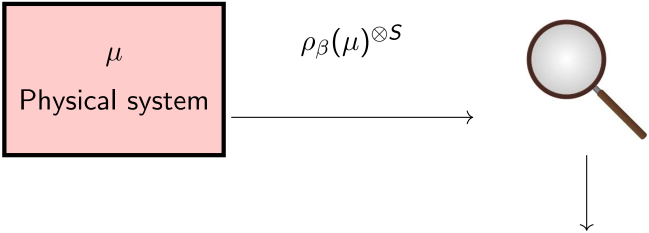
- Why these 3 classes of states?
  - They capture all three range in our understanding: fairly well understood, mildly well understood, very unclear.
  - Fundamental quantum many-body ansatz in the regime of low entanglement.

Upcoming section

Learning unknown Hamiltonian from Gibbs state



μ' such that $\forall i, |\mu'_i - \mu_i| \leq \varepsilon$

Recall: $H(\mu) = \sum_i \mu_i E_i$.

Main open question

Is there a polynomial time learning algorithm that achieves this task with sample complexity

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The known lower bound is

$$\Omega\left(\log(m) \cdot \exp(O(\beta)) \cdot \frac{1}{\varepsilon^2}\right).$$

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- (A., Arunachalam, Kuwahara, Soleimanifar 2020) - non-negligible changes in local data force the Hamiltonians to be different. A $O(\mathrm{poly}(n) e^{\mathrm{poly}(\beta)}/\varepsilon^2)$ sample complexity can be obtained.

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- Main technical ingredient: strong convexity of quantum partition function. Variance lower bound on extensive observables.

Time efficient algorithm

- (Bakshi, Liu, Moitra, Tang, 2024): $O(\mathrm{poly}(n)e^{e^{\mathrm{poly}(\beta)}}/\varepsilon^2)$ sample and time complexity.
- Entangling measurements act on $e^{\mathrm{poly}(\beta)}$ qubits.
- Recent improvement: $e^{\mathrm{poly}(\beta)} \to \mathrm{poly}(\beta)$.
- New locality property of the imaginary time evolution operator $e^{\beta H} A e^{-\beta H}$.
- Sum-of-squares techniques play a crucial role.

Known results

- Optimal algorithms known for
  - (Haah, Kothari, Tang, 2023): High temperature case ($\beta < \beta_c$).
  - (A., Arunachalam, Kuwahara, Soleimanifar, 2021): Commuting case.
- Learning Gibbs states in quantum Wasserstein distance is also considered in (Rouze, Franca, Quantum 2024) and (Palma, Marvian, Trevisan, Lloyd, 2021).

Open directions

- Improving $\mathrm{poly}(n)$ sample complexity to $\mathrm{poly}\log n$ sample complexity.

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- Learning the structure of the interaction graph, with sparsity the only guarantee (already open for the commuting case).

Upcoming section

Motivation and goals

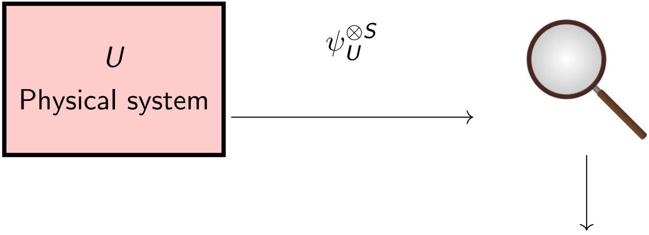
Learning Gibbs state

Shallow quantum states

Tensor network states

Learning shallow quantum states

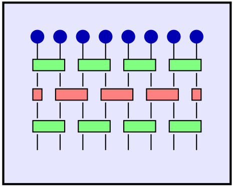
- Quantum states of the form $|\psi_U\rangle = U|0^n\rangle$, where U is a quantum circuit of depth t, have emerged as important ansatz in approximating low energy quantum states.



W such that $\| W|0^m\rangle - |\psi_U\rangle |junk\rangle \|_1 \leq \varepsilon$

Sample efficient learning

Local data: marginals $\psi_{U,i}$ of the qubits in the 'light cones' L_i (set of qubits that can influence the qubit i).



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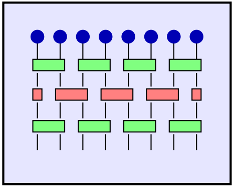

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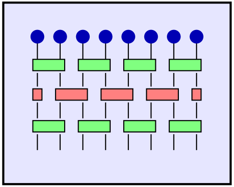

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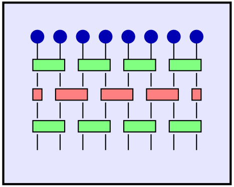

- Search over all unitaries U' of depth t, compute the marginals of $\psi_{U'}$ and see if they are close.
- If $\psi_{U',i} \overset{\varepsilon/n}{\approx} \psi_{U,i}$, then $\psi_{U'} \overset{\varepsilon}{\approx} \psi_U$.
- Thus, sample complexity is $2^{O(t)} \frac{n^2}{\varepsilon^2}$.

Known results

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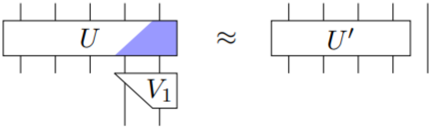
- (Liu, Landau, 24) developed an algorithm that works in higher dimensions.
- (Huang, Liu, Broughton, Kim, A., Landau, McClean, STOC 24) also solve the problem with "circuit" access - given a blackbox that implements U, output a W that's close to to U in diamond norm.
    - W has depth $O(t)$ on finite dimensional lattices. Depth is $2^{O(t)}$ on general graphs.

Known results

- These algorithms work by finding "local inversions" to the circuit and then developing schemes to join them together.
- The measurements act on qubits roughly the size of the 'lightcone'.
- Technical contribution goes in resolving the following issue - local inversions may not be consistent with each other.



Open directions

- Can shallow state learning be achieved on general graphs?
- Can the depth of output circuit in circuit learning be optimized to $O(t)$ on general graphs?
- Can the sample complexity be significantly improved in the presence of noise?

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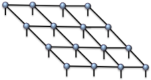

$\vec{A}$

Physical system

$\psi_{\vec{A}}^{\otimes S}$ $\longrightarrow$

$\downarrow$

$\vec{A}'$ such that $\| \, |\psi_{\vec{A}'}\rangle - |\psi_{\vec{A}}\rangle \, \|_1 \leq \varepsilon$

Are tensor network states locally learnable?

- Consider the CAT states:

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- Consider the CAT states:

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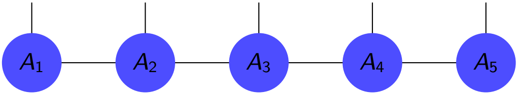
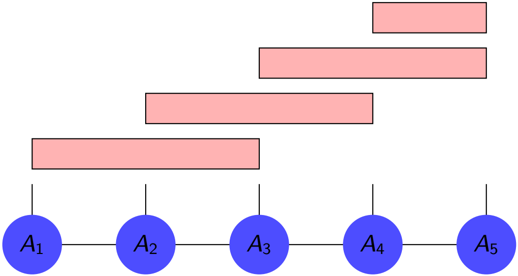
- These can be written as 1D Tensor Network States (Matrix Product States)



- However, they look the same locally! Thus, they can't be learned locally in general.

Learning Matrix Product States

- Matrix product states (with matrices of dimension D) can be written as a sequential circuit: $|\psi\rangle = U_1 U_2 \ldots U_{n-1} |0^n\rangle$.



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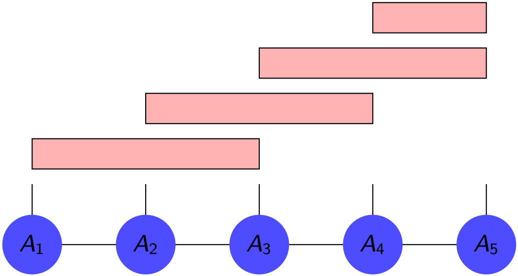

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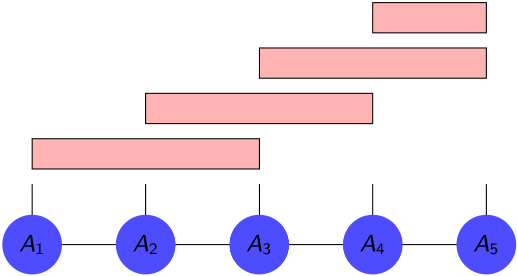

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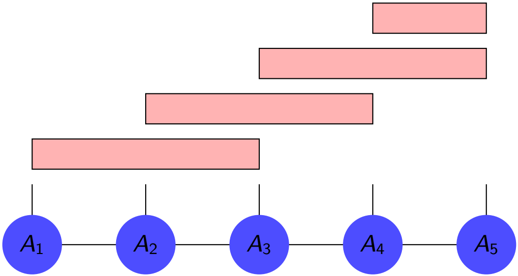

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- Does not work for general tensor networks.

Locally learning classes of Matrix Product States

- Main open question - obtain efficient learning algorithm using local measurements, which outputs an MPS of bond dimension $O(D)$.
    - Assuming certain "invertability condition".
- Progress in [Fanizza, Galke, Lumbreras, Rouze, Winter, 2023], which learns a Matrix Product Operator of bond dimension D^2 that is close to the MPS.

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- However, no time efficient method is known.
- Technical question that might help - for a tensor network with gapped parent Hamiltonian, is it possible to reconstruct the tensor network ground state?

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- Thanks for your attention!