Solutions of the Stochastic Dirichlet Problem

Jerry Li

June 20, 2013

Contents

1 Preliminaries

1.1 Continuous Probability

We assume the reader is fluent in basic measure theory and has some familiarity with continuous probability, so we cover this very quickly in a quick parade of definitions. Fix some measurable space $(\Omega, \mathcal{F})$. A *probability measure* on Ω is a measure P so that $P(\Omega) = 1$, and we call $(\Omega, \mathcal{F}, P)$ a *probability space*. Throughout the paper there will be some tacit space Ω equipped with a probability measure P, and we let expectation with respect to P be denoted by $\mathbb{E}$. When it is clear from context we will let $\Pr(A) = P(A)$, as is classical.

A *random variable* X is a $\mathbb{R}^n$-valued measurable function with domain Ω, and a *stochastic process* is a set of random variables $\{X_t\}_{t\in S}$ indexed by $t \in S$, which we will think of as a time variable. For us, S will usually be some subinterval of the real line. As usual, for all stochastic process $X_t(\omega)$, we will interchangeably use the notations $X_t(\omega)$ and $X(t, \omega)$.

Let S be totally ordered by $\leq$ (for instance, if S is an interval or $S = \mathbb{N}$). A *filtration* is a set of σ-fields $\{\mathcal{F}_t\}_{t\in S}$ so that $\mathcal{F}_t \subset \mathcal{F}$ for all $t \in S$, and if $s \leq t$ then $\mathcal{F}_s \subseteq \mathcal{F}_t$. We let $\mathcal{F}_\infty$ be the smallest σ-algebra containing every $\mathcal{F}_t$. We say that a stochastic process X_t is $\mathcal{F}_t$ *adapted* if X_t is $\mathcal{F}_t$ measurable for all $t \in S$. A *stopping time* $\tau : \Omega \to S$ is a function so that $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \in S$. For any stopping time τ and stochastic process X_t we let $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$ and we let

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \in S\}.$$

The author has asked a number of professors what this σ-algebra means intuitively, and has been told to stop thinking about it, so he advises any readers to also do so. Apparently, this is what makes the results work.

If $\mathcal{G} \subseteq \mathcal{F}$ is another σ-algebra, and X is any random variable, we let $\mathbb{E}[X|\mathcal{G}]$ denote the unique $\mathcal{G}$-measurable random variable Y so that $\mathbb{E}[X 1_A] = \mathbb{E}[Y 1_A]$ for all $A \in \mathcal{G}$. It is well-known that this is well-defined ([1], section 5.1), and we call Y the *expectation of* X *conditioned on* $\mathcal{G}$. Finally, if X_t is a $\mathcal{F}_t$-adapted stochastic process, we say X_t is a *martingale* with respect to $\mathcal{F}_t$ if for all $s \leq t$, $\mathbb{E}[X_t|\,\mathcal{F}_s] = X_s$. Martingales have a myriad of very important properties. We list two which will be very important for us.

Theorem 1.1 (Martingale convergence theorem, [2] p. 236)**.** *Let X_t be a martingale with $S = \mathbb{N}$ or $S = \mathbb{R}_{\geq 0}$ with respect to $\mathcal{F}_t$, for $t \in S$. If $\sup_{t\in S} \mathbb{E}[|X_t|^p] < \infty$ then X_t converges almost surely and in L^p to some random variable X. In particular, if the X_t are uniformly bounded, $X_t \to X$ almost surely and in L^p for all $p \geq 0$.*

Theorem 1.2 (Doob's martingale inequality, [2] p. 249)**.** *Let X_t be a martingale with respect to $\mathcal{F}_t$, for $t \in S$. If $S = \mathbb{N}$, then*

$$\Pr\left[\sup_{0\leq t\leq T} X_t \geq C\right] \leq \frac{\mathbb{E}[X_T^p]}{C^p},\ \forall p \geq 1. \tag{1.1}$$

If $S = \mathbb{R}_{\geq 0}$ and for almost all ω, the function $t \mapsto X(t, \omega)$ is right-continuous and has left limits, then (1.1) holds.

1.2 Brownian Motion

We again assume the reader has some familiarity with continuous probability. We tacitly fix some probability space $(\Omega, \mathcal{F}, P)$. For our purposes it suffices to think of $\Omega = C([0, \infty))$, the space of continuous real-valued functions on $[0, \infty)$, with the natural σ-algebra generated by finite projections.

Definition 1.1. A *Brownian motion* is a stochastic process $B_t(\omega)$ so that

1. $\Pr(\{\omega : t \mapsto B_t(\omega)\}$ is continuous$) = 1$; that is, B_t has almost surely continuous paths.
2. If $t > s \geq 0$, then $B_t - B_s$ is independent of B_s.
3. If $t > s \geq 0$ then $B_t - B_s \sim N(0, t - s)$.

We also define n-dimensional Brownian motion to be a process which is n independent copies of one-dimensional Brownian motion.

The following theorems can all be found with proof in [2], Ch. 8.

Theorem 1.3. *There exists a probability space $(\Omega, \mathcal{F}, P)$ (in fact $\Omega = C([0,\infty))$, with the natural σ-algebra) on which for all $x \in \mathbb{R}$ there exists an almost surely unique Brownian motion B_t so that $B_0 = x$.*

We let $\mathcal{F}_t^0$ be the σ-algebra generated by B_s for $s \leq t$. In the normal procedure, we let $\mathcal{F}_t$ be the standard completion and augmentation of $\mathcal{F}_t$. We let B_t^x denote the unique Brownian motion with $B_0 = x$.

Theorem 1.4.

1. *Brownian motion is a martingale with respect to $\mathcal{F}_t$; that is, $\mathbb{E}[B_t|\mathcal{F}_s] = B_s$ for all $t > s \geq 0$.*
2. *Brownian motion is a strong Markov process (and in particular a Markov process); i.e., for all stopping times τ and for all $f : \mathbb{R} \to \mathbb{R}$ Borel measurable,*

$$\mathbb{E}[f(B_{\tau+h}^x)|\,\mathcal{F}_\tau](\omega) = \mathbb{E}[f(X_h^y)]|_{y=X_\tau^x(\omega)}.$$

We ask now how nice paths of Brownian motion are, for eventually we wish to integrate over them. We know that the paths are continuous, however, it turns out that they are almost surely not differentiable ([2], p. 358). Moreover, they do not have finite length (almost surely).

Definition 1.2. Let k be a positive integer, and $T > 0$. For any function $g : [0,T] \to \mathbb{R}$, if the quantity

$$S_k(g,T) = \lim_{|\Delta|\to 0} \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})|^k$$

exists, then we call it the kth-variation of g, where $\Delta = \{t_0, \ldots, t_n\}$ is a partition of T, and $|\Delta| = \max_n |t_i - t_{i-1}|$.

Theorem 1.5. *For all $T > 0$,*

$$\Pr[\omega : S_2(B(t,\omega),T) = T] = 1.$$

In particular, Brownian motion has infinite first variation and zero kth variation for $k > 2$.

This poses a significant challenge to integration along the paths of Brownian motion, as normal Riemann-Stieltjes integration is defined only over rectifiable paths, namely, paths with finite first variation. This justifies the need for a more advanced theory of integration.

2 Definition of Ito Integrals

2.1 Motivation and Construction

The discussion here closely follows the discussion in [5]. The first requirement for understanding solutions to stochastic Dirichlet problems is getting a notion of a stochastic calculus, which is where Ito integrals come into play. They allow us to understand the fairly abstract quantity

$$(If)(\omega) = \int_S^T f(t,\omega)dB_t$$

where B_t is a standard Brownian motion. A priori, such a quantity is not well-defined, as Brownian motion almost surely has finite quadratic variation and infinite finite variation, so traditional Lebesgue–Stieltjes integration path-wise is not well-defined.

Given a function $f(t,\omega)$ defined for $S \le t \le T$, a natural approximation is given by functions of the form

$$\sum_j f(t_j^*, \omega)\chi_{(t_j, t_{t+1}]}(t)$$

where the t_j form a partition of $[S,T]$, and these approximations form the basis of Lebesgue–Stieltjes integrals. However, when we attempt to integrate along Brownian motion, because Brownian motion is almost surely not rectifiable the choice of t_j^* matters in the limit. Ito integrals correspond to the choice $t_j^* = t_j$; that is, the left endpoint; this is a reasonable (but not the only) choice for reasons we will discuss later.

The formal definition of the Ito integral ought to be familiar to the reader. We first define the integral over the stochastic equivalent of step functions; then approximate a more general class of functions with these step functions. Let $\mathcal{F}_t$ be any filtration so that B is a Brownian motion with respect to $\mathcal{F}_t$. Usually we can let $\mathcal{F}_t$ be the natural filtration generated by the Brownian motion however in some important applications we will need it to be the usual augmented filtration.

Definition 2.1. A function $\phi : [0,\infty) \times \Omega \to \mathbb{R}$ is elementary if it can be written as

$$\phi(t,\omega) = \sum_j e_j(\omega)\chi_{(t_j, t_{j+1}]}(t)$$

where the $\{t_j\}_{j=0}^{\infty}$ form a partition of $[0,\infty)$ and each e_j is $\mathcal{F}_{t_j}$ measurable.

For any elementary function ϕ we define the Ito integral $I\phi$ to be

$$I\phi(\omega) = \int_S^T \phi(t,\omega)dB_t = \sum_j e_j(\omega)\left[B_{t_{j+1}^*} - B_{t_j^*}\right](\omega).$$

where $t_j^* = t_j$ if $S \le t_j \le T$ and

$$t_j^* = \begin{cases} t_j & \text{if } S < t_j < T, \\ S & \text{if } t_j \le S, \\ T & \text{if } t_j \ge T \end{cases}$$

since in the integral we want only to consider the differences that occur between S and T. We now seek to extend this definition to a wider class of functions. We do so with the help of the following, very important isometry.

Theorem 2.1 (The Ito Isometry for elementary functions)**.** *If ϕ is bounded and elementary then*

$$\mathbb{E}\left[\left(\int_S^T \phi(t,\omega)dB_t(\omega)\right)^2\right] = \mathbb{E}\left[\int_S^T \phi(t,\omega)^2 dt\right].$$

Proof. Write $\phi(t,\omega) = \sum_j e_j(\omega)\chi_{(t_j,t_{j+1}]}(t)$, and let $\Delta B_j = B_{t_{j+1}^*} - B_{t_j^*}$. Then

$$\begin{aligned} \mathbb{E}\left[\left(\int_S^T \phi(t,\omega)dB_t(\omega)\right)^2\right] &= \sum_{i,j}\mathbb{E}[e_i e_j \Delta B_i \Delta B_j] = \sum_i \mathbb{E}[e_i^2](t_{i+1} - t_i) \\ &= \mathbb{E}\left[\int_S^T \phi(t,\omega)^2 dt\right] \end{aligned}$$

as if $i < j$ we have $\mathbb{E}[e_i e_j \Delta B_i \Delta B_j] = 0$ since $e_i e_j \Delta B_i$ is independent of ΔB_j. □

Definition 2.2. Let $\mathcal{N}(S,T)$ be the class of functions $\mathcal{B} \times \mathcal{F}$ measurable functions $f : [0,\infty) \times \Omega \to \mathbb{R}$ so that $f(t,\omega)$ is $\mathcal{F}_t$ adapted and

$$\mathbb{E}\left[\int_S^T f(t,\omega)^2 dt\right] < \infty.$$

We say $f \in \mathcal{N}$ if $f \in \mathcal{N}(0,T)$ for all $T \geq 0$.

We intend to define the Ito integral for all functions in $\mathcal{N}$. To do so, we will need to following lemma.

Lemma 2.2. *For all T, and for every $f \in \mathcal{N}(S,T)$ there exists a sequence of elementary ϕ_n so that*

$$\mathbb{E}\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 dt\right] \to 0. \tag{2.1}$$

Proof. We will do so in several steps. First, we show that we can approximate bounded, continuous functions with elementary functions, then approximate bounded functions with bounded, continuous functions, and finally functions in $\mathcal{N}(S,T)$ with bounded functions.

First, let f be bounded and continuous for each ω. Then if we define $\phi_n(t,\omega) = \sum_j g(t_j,\omega)\chi_{(t_j,t_{j+1}]}$. By continuity and the dominated convergence theorem we get (2.1). Now, let f be bounded. For each n, let ϕ_n be a non-negative continuous function whose support is $[-1/n, 0]$ and $\int_{-\infty}^{\infty} \phi_n dx = 1$. Let

$$g_n(t,\omega) = \int \phi_n(s-t) f(s,\omega) ds$$

be the convolution of ϕ with f. It is not hard to show that for fixed ω this is continuous, and by the $\mathcal{F}_t$ measurability of $f(t,\omega)$ for all t, it is readily shown that $g_n(t,\cdot)$ is $\mathcal{F}_t$ measurable as well. It is well known that the convolutions with these nascent Dirac delta functions converges to f in L^1 and hence in L^2 as f and g_n are both bounded so (2.1) is satisfied.

Finally, if $f \in \mathcal{S}, \mathcal{T}$ we let g_n be the truncation of f at n; that is, $g_n(t,\omega) = f(t,\omega)$ if $|f(t,\omega)| \leq n$, otherwise, $g_n(t,\omega) = n$. By a final application of the dominated convergence theorem we are done. □

Hence for each $f \in N(S,T)$ we may choose a sequence ϕ_n of elementary functions so that (2.1) holds; then define

$$\int_S^T f(t,\omega) dB_t = \lim_{n\to\infty} \int_S^T \phi_n dB_t.$$

The ϕ_n are Cauchy in L^2 by Theorem 2.3 and hence such a limit exists; by Theorem 2.3 it is also clear that this limit is independent of the choice of ϕ_n and Ito's isometry holds for f. This is sufficiently important so we repeat it below.

Theorem 2.3 (The Ito Isometry)**.** *For all $f \in N$ then*

$$\mathbb{E}\left[\left(\int_s^t f(t,\omega) dB_t(\omega)\right)^2\right] = \mathbb{E}\left[\int_s^t f(t,\omega)^2 dt\right].$$

2.2 Properties of the Ito Integral

Proposition 2.4. *Suppose $f, g \in \mathcal{N}(S,T)$ and $u \in (S,T)$. Then*

- $\int_S^T af + bg dB_t = a\int_S^T f dB_t + b\int_S^T g dB_t,$ *a.s.*
- $\int_S^T f dB_t = \int_S^u f dB_t + \int_u^T f dB_t,$ *a.s.*

- $\mathbb{E}[\int_s^t f dB_t] = 0$
- *The process* $I_t = \int_s^t f dB_t$ *is a martingale.*

Proof. All these properties are easily checked to be true for f elementary and are preserved under L^2 limits. $\square$

Remark 2.1. In the proof of that the integral of an elementary function is a martingale the reader may check that it is vital that we choose the left endpoint in the definition of the Ito integral; for any other choice of an endpoint this fails. Because being a martingale is a very strong property, this is a very strong case for using the left endpoint.

Given a stochastic process X_t, where $t \in [0, \infty)$, we say that X_t admits a *continuous modification* if there exists a process Y_t so that $\Pr(X_t = Y_t) = 1$ for all $t \in [0, \infty)$, and the function $t \mapsto Y_t(\omega)$ is continuous for almost all ω. Doob's martingale inequality can now be used to prove that $\int_s^t f dB_t$ admits a continuous modification:

Theorem 2.5. *Fix* $f \in \mathcal{N}[0, T]$. *Then there exists a stochastic process* X_t *with* t*-continuous paths so that*

$$\Pr\left[X_t = \int_0^t f(s) dB_s\right] = 1, \forall t \in [0, \infty).$$

Proof. Let ϕ_n be elementary so that

$$\mathbb{E}\left[\int_0^t (f - \phi_n)^2 \, dt\right] \to 0$$

and define

$$I_n(t, \cdot) = \int_0^t \phi_n dB_s$$

and

$$I(t, \cdot) = \int_0^t f dB_s$$

where we define $\int_0^S f dB_s = \int_0^T f dB_s$ for $S \geq T$. By Proposition 2.4 these processes are martingales; in particular $I_n - I_m$ is a martingale, so by Doob's martingale inequality

$$\begin{aligned}
\Pr\left[\omega : \sup_{0 \leq t \leq T} |I_n(t, \omega) - I_m(t, \omega)| > \epsilon\right] &\leq \frac{1}{\epsilon^2} \mathbb{E}\left[|I_n(T, \omega) - I_m(t, \omega)|^2\right] \\
&= \frac{1}{\epsilon^2} \mathbb{E}\left[\int_0^T |\phi_n(s, \omega) - \phi_m(s, \omega)|^2 ds\right] \to 0
\end{aligned}$$

as $n, m \to \infty$ by our choice of ϕ_n.

Choose m_1, m_2 so that for all $n_1 \geq m_1$ and $n_2 \geq m_2$ we have

$$\Pr\left[\omega : \sup_{0 \leq t \leq T} |I_{n_1}(t, \omega) - I_{n_2}(t, \omega)| > 2^{-1}\right] < 2^{-1}.$$

Set $n_1 = m_1$, and inductively, given n_{k-1} and m_k so that for all $n_k \geq m_k$

$$\Pr\left[\omega : \sup_{0 \leq t \leq T} |I_{n_{k-1}}(t, \omega) - I_{n_k}(t, \omega)| > 2^{-(k-1)}\right] < 2^{-(k-1)}$$

we choose $m_k' \geq m_k$ and $m_{k+1} \geq m_k'$ so that

$$\Pr\left[\omega : \sup_{0\leq t\leq T} |I_{n_k}(t,\omega) - I_{n_{k+1}}(t,\omega)| > 2^{-k}\right] < 2^{-k}$$

for all $n_k \geq m_k'$. Set $n_k = m_k'$ and inductively we see that we have constructed a subsequence $\{n_k\}$ so that

$$\Pr\left[\omega : \sup_{0\leq t\leq T} |I_{n_k}(t,\omega) - I_{n_{k+1}}(t,\omega)| > 2^{-k}\right] < 2^{-k}$$

for all $k \geq 0$. By Borel-Cantelli,

$$\Pr\left[\omega : \sup_{0\leq t\leq T} |I_{n_k}(t,\omega) - I_{n_{k+1}}(t,\omega)| > 2^{-k} \text{ i.o.}\right] = 0$$

so for almost all ω the $I_{n_k}(t,\omega)$ converge uniformly in t and ω to some value J_t. As each $I_{n_k}(t,\omega)$ is continuous, so is J_t, and furthermore, since $I_{n_k}(t,\omega) \to I(t,\omega)$ in L^2 for all fixed t we conclude that $J_t = I_t$ for all t and for almost all ω, and we are done. □

Remark 2.2. It is possible to define Ito integrals on an even larger space of functions; namely, functions f suitably adapted to the correct filtration where $\Pr\left[\int_0^t |f(t,\omega)|^2 dt < \infty, \forall t \geq 0\right] = 1$. In doing so our convergence goes from convergence in L^2 to convergence in probability; we do not go into specifics here ([5], p. 26). It is also possible to define Ito integrals for all local semi-martingales ([3], Ch. 3), instead of just Brownian motion (although by the martingale representation theorem this is not as strong as it would seem a priori); in this context Ito's formula admits a nice, general form and a much cleaner proof.

2.3 Multi-dimensional Ito Integrals

Let $B = (B_1, \ldots, B_n)$ be a n-dimensional Brownian motion and for $t \in [s,t]$ let $v(t,\omega) = [v_{ij}(t,\omega)]_{ij}$ be a $m \times n$ matrix where each v_{ij} is $\mathcal{B} \times \mathcal{F}$-measurable and satisfies

$$\mathbb{E}\left[\int_s^t v_{ij}(\zeta,\omega)^2 d\zeta\right] < \infty \tag{2.2}$$

and v is adapted to $\mathcal{F}_t^{(n)}$ where $\mathcal{F}_t^{(n)}$ is generated by $B_k(s,\cdot)$ for $k = 1, \ldots, n$ and $s \leq t$. It is easy to check that B is a Brownian motion with respect to this filtration, so we may define

$$\left(\int_s^t v dB\right)_i = \sum_{j=1}^n \int_s^t v_{ij}(t,\omega) dB_j.$$

3 Ito's Formula

Suppose $v \in N(0,T)$ with respect to the filtration $\mathcal{F}_t$ where B is a Brownian motion with respect to $\mathcal{F}_t$ as well, and let $u(t,\omega)$ be a $\mathcal{F}_t$ adapted stochastic process, and X_0 is some $\mathcal{F}$-measurable function. Then if

$$X_t = X_0 + \int_0^t u(s,\omega) ds + \int_0^t v(s,\omega) dB_s \tag{3.1}$$

we say that X is a (one-dimensional) stochastic integral. In the literature, this is sometimes written in the differential form

$$dX_t = u dt + v dB_t$$

where this is simply understood to mean that X_t satisfies (3.1).

While we know that these objects are well-defined, a priori it is hard to do even simple calculations with them. To our aid comes Ito's formula, stated below for the one-dimensional case. We actually state and prove a weaker version of the theorem.

Theorem 3.1 (Ito's formula in one dimension)**.** *Let $v \in \mathcal{N}(0,T)$ with respect to $\mathcal{F}_t$ and let u be $\mathcal{F}_t$ adapted, where B is a Brownian motion with respect to $\mathcal{F}_t$. Suppose also that*

$$\Pr\left[\int_0^t u(s,\omega)ds < \infty\right] = 1.$$

Let X_t be so that

$$dX_t = udt + vdB_t.$$

Then if $g(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is C^2, and

$$\sup_{0 \le t \le T, x \in \mathbb{R}} |g(t,x)| < \infty$$

for all T, then $Y_t = g(t, X_t)$ is also a stochastic integral with representation

$$Y_t = g(0,X_0) + \int_0^t \frac{\partial g}{\partial x} dB_s + \int_0^t \left[\frac{\partial g}{\partial t} + u\frac{\partial g}{\partial x} + \frac{1}{2}v^2 \frac{\partial^2 g}{\partial x^2}(s, X_s)\right] ds, \tag{3.2}$$

where the functions are being evaluated at (s, X_s). Alternatively, in the differential notation, letting $dX_t = udt + vdB_t$

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial t}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(s, X_s)(dX_t)^2.$$

where we let $d_t \cdot d_t = d_t \cdot dB_t = dB_t \cdot d_t = 0$ and $dB_t \cdot dB_t = t$.

Remark 3.1. It is possible to remove the boundedness condition on g; however to make sense of that integral requires the extension of Ito integrals to a larger class of functions, which we neglect to do. See Remark 2.2.

Proof. For fixed t, the functions $g, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}, \frac{\partial^2 g}{\partial x^2}$ are bounded. Suppose u and v are elementary. Let $0 = t_1 < t_2 < \ldots < t_k = t$ be a partition of $[0,t]$, and let $\Delta t_j = t_{j+1} - t_j, \Delta B_j = B_{t_{j+1}} - B_{t_j}, \Delta X_j = X_{t_{j+1}} - X_{t_j}$, and $\Delta g(t_j, X_j) = g(t_{j+1}, X_{t_{j+1}}) - g(t_j, X_{t_j})$. Then by Taylor's theorem since

$$\Delta(g(t_j, X_{t_j})) = \frac{\partial g}{\partial t}(t_j, X_{t_j})\Delta t_j + \frac{\partial g}{\partial x}(t_j, X_{t_j})\Delta X_j + \frac{\partial^2 g}{\partial t \partial x}\Delta t_j \Delta X_j + \frac{1}{2}\frac{\partial^2 g}{\partial t^2}\Delta t_j^2 + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\Delta X_j^2 + R_j$$

where $R_j = O(|\Delta t_j|^3 + |\Delta X_j|^3)$, we get that

$$\begin{aligned} g(t, X_t) &= g(0,X_0) + \sum_j \Delta g(t_j, X_j) \\ &= \sum_j \frac{\partial g}{\partial t}(t_j, X_{t_j})\Delta t_j + \sum_j \frac{\partial g}{\partial x}(t_j, X_{t_j})\Delta X_j + \sum_j \frac{\partial^2 g}{\partial t \partial x}\Delta t_j \Delta X_j \\ &+ \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial t^2}\Delta t_j^2 + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial x^2}\Delta X_j^2 + \sum_j R_j. \end{aligned}$$

We now consider how each sum behaves as we take partitions so that $\max_j |\Delta t_j| \to 0$. It is clear that

$$\sum_j \sum_j \frac{\partial g}{\partial t}(t_j, X_{t_j})\Delta t_j \to \int_0^t \frac{\partial g}{\partial t} s, X_s ds$$

for each ω, as for ω fixed both sides are simply normal Riemann sums, and convergence in L^2 occurs because the function $\frac{\partial g}{\partial t}$ is bounded.

Let $u = \sum_k e_k(\omega)\chi_{(T_k,T_{k+1}]}(t)$ and $v = \sum_k d_k(\omega)\chi_{(S_k,S_{k+1}]}(t)$ and take refinements of our partitions so that the points T_k and S_k are always included in the partition. Let s_k be the entwinement of T_k and S_k. Then for t_j, t_{j+1} between s_k and s_{k+1} we have $\Delta X_= e_{k_e(k)}(t_{j+1}-t_j) + d_{k_d(s)}(B_{t_{j+1}} - B_{t_j})$ where $k_e(k) = k'$ if $s_k \leq T_{k'} < s_{k+1}$ and similarly for k_d. Hence,

$$\begin{aligned}\sum_{j:s_k\leq t_j<s_{k+1}} \sum_j \frac{\partial g}{\partial x}(t_j, X_{t_j})\Delta X_j &= \sum_{j:s_k\leq t_j<s_{k+1}} \sum_j \frac{\partial g}{\partial x}(t_j, X_{t_j})e_{k_e(k)}(t_{j+1}-t_j) \\ &+ \sum_{j:s_k\leq t_j<s_{k+1}} \sum_j \frac{\partial g}{\partial x}(t_j, X_{t_j})d_{k_d(k)}(B_{t_{j+1}} - B_{t_j}).\end{aligned}$$

Again it is clear that pointwise and in L^2

$$\sum_{j:s_k\leq t_j<s_{k+1}} \sum_j \frac{\partial g}{\partial x}(t_j, X_{t_j})e_{k_e(k)}(t_{j+1}-t_j) \to \int_{s_k}^{s_{k+1}} u(s,\omega)\frac{\partial g}{\partial x}(s, X_s)ds$$

and from the definition of Ito integrals it is evident that

$$\sum_{j:s_k\leq t_j<s_{k+1}} \sum_j \frac{\partial g}{\partial x}(t_j, X_{t_j})d_{k_d(k)}(B_{t_{j+1}} - B_{t_j}) \to \int_{s_k}^{s_{k+1}} v(s,\omega)\frac{\partial g}{\partial x}(t_j, X_{t_j})dB_s$$

in L^2, so by linearity of the regular integral and Ito's integral we have that

$$\sum_j \frac{\partial g}{\partial t}(t_j, X_{t_j})\Delta t_j + \sum_j \frac{\partial g}{\partial x}(t_j, X_{t_j})\Delta X_j \to \int_0^t \frac{\partial g}{\partial t}ds + \int_0^t \frac{\partial g}{\partial x}dB_s.$$

We next show that

$$\frac{\partial^2 g}{\partial t \partial x}\Delta t_j \Delta X_j \to 0$$

in basically the same manner. We write

$$\begin{aligned}\sum_{j:s_k\leq t_j<s_{k+1}} \frac{\partial^2 g}{\partial t\partial x}(t_j, X_{t_j})\Delta t_j\Delta X_j &= \sum_{j:s_k\leq t_j<s_{k+1}} \frac{\partial g}{\partial x}(t_j, X_{t_j})e_{k_e(k)}(t_{j+1}-t_j)^2 \\ &+ \sum_{j:s_k\leq t_j<s_{k+1}} \frac{\partial g}{\partial x}(t_j, X_{t_j})d_{k_d(k)}(t_{j+1}-t_j)(B_{t_{j+1}} - B_{t_j}).\end{aligned}$$

But

$$|\sum_{j:s_k\leq t_j<s_{k+1}} \frac{\partial g}{\partial x}(t_j, X_{t_j})e_{k_e(k)}(t_{j+1}-t_j)^2| \leq \max_j \Delta t_j \sum_{j:s_k\leq t_j<s_{k+1}} |\frac{\partial g}{\partial x}(t_j, X_{t_j})e_{k_e(k)}|(t_{j+1}-t_j) \to 0$$

pointwise as $\sum_j |\frac{\partial g}{\partial x}(t_j, X_{t_j})e_{k_e(k)}|(t_{j+1}-t_j)$ converges to a finite Riemann integral and $\max_j \Delta t_j \to 0$; convergence in L^2 follows because all quantities on the right side are bounded. By basically the same argument,

$$\sum_{j:s_k\leq t_j<s_{k+1}} \frac{\partial g}{\partial x}(t_j, X_{t_j})d_{k_d(k)}(t_{j+1}-t_j)(B_{t_{j+1}} - B_{t_j}) \to 0$$

in L^2, so by linearity

$$\frac{\partial^2 g}{\partial t \partial x}\Delta t_j \Delta X_j \to 0,$$

as claimed. Using the same argument again for the next term we get that

$$\frac{1}{2}\sum_j \frac{\partial^2 g}{\partial t^2}\Delta t_j^2 \to 0$$

almost surely and in L^2. Because Brownian motion has finite second variation (and hence zero third variation), the remainder term also goes to zero; therefore it suffices to check that

$$\sum_j \frac{\partial^2 g}{\partial x^2}\Delta X_j^2 \to \int v^2 \frac{\partial^2 g}{\partial x^2}(s, X_s)ds.$$

Again, we split up the left sum with the s_j, and we write

$$\sum_{s_k \le t_j < s_{k+1}} \frac{\partial^2 g}{\partial x^2}\Delta X_j^2 = \sum_{s_k \le t_j < s_{k+1}} \frac{\partial^2 g}{\partial x^2} d_{k_d(e)}^2 \Delta B_j^2 + 2\sum_{s_k \le t_j < s_{k+1}} \frac{\partial^2 g}{\partial x^2} e_{k(e)} d_{k(d)} \Delta t_j \Delta B_j + \sum_{s_k \le t_j < s_{k+1}} \frac{\partial^2 g}{\partial x^2} e_{k(e)}^2 \Delta t_j^2.$$

We showed previously that the last two terms go to zero in L^2 as the partition grows finer, so it suffices to show that

$$\sum_{s_k \le t_j < s_{k+1}} \frac{\partial^2 g}{\partial x^2}\Delta B_j^2 \to \int v^2 \frac{\partial^2 g}{\partial x^2}(s, X_s)ds$$

in L^2.

We claim that

$$\mathbb{E}\left[\left(\sum_{s_k \le t_j < s_{k+1}} \frac{\partial^2 g}{\partial x^2} d_{k_d(k)} \Delta B_j^2 - \sum_{s_k \le t_j < s_{k+1}} \frac{\partial^2 g}{\partial x^2} d_{k_d(k)} \Delta t_j\right)^2\right] \to 0$$

as $|\max_j \Delta t_j| \to 0$; as $\sum_{s_k \le t_j < s_{k+1}} \frac{\partial^2 g}{\partial x^2}\Delta t_j \to \int v^2 \frac{\partial^2 g}{\partial x^2}(s, X_s)ds$ pointwise and in L^2 (as the functions are bounded) this is sufficient. Let $a(t) = \frac{\partial^2 g}{\partial x^2}(t, X_t) d_{k_d(k)}$ and $a_j = a(t_j)$. Then the expectation is simply

$$\mathbb{E}\left[\left(\sum_{s_k \le t_j < s_{k+1}} a_j \Delta B_j^2 - \sum_{s_k \le t_j < s_{k+1}} a_j \Delta t_j\right)^2\right] = \sum_{s_k \le t_i, t_j < s_{k+1}} \mathbb{E}\left[\left(a_i a_j (\Delta B_j^2 - \Delta t_j)(\Delta B_i^2 - \Delta t_i)\right)\right].$$

But if $i < j$ then $a_i a_j (\Delta B_i^2 - \Delta t_j)$ is independent of $\Delta B_j^2 - \Delta t_j$ so as $E(\Delta B_j^2 - \Delta t_j) = 0$ (by the definition of Brownian motion) those terms do not contribute anything to the sum. Therefore as a_j is independent from ΔB_j the expectation is exactly

$$\sum_{s_k \le t_i < s_{k+1}} \mathbb{E}\left[\left(a_i (\Delta B_i^2 - \Delta t_i)^2\right)\right] = \sum_{s_k \le t_i < s_{k+1}} \mathbb{E}(a_i^2)\, \mathbb{E}\left[\Delta B_i^4 - 2\Delta B_i^2 \Delta t_i + \Delta t_i^2\right].$$

As the fourth moment of a Gaussian variable with variance σ^2 is $3\sigma^4$ we conclude that

$$\mathbb{E}\left[\Delta B_i^4 - 2\Delta B_i^2 \Delta t_i + \Delta t_i^2\right] = 3\Delta t_i^2 - 2\Delta t_i^2 + \Delta t_i^2 = 2\Delta t_i^2$$

so the entire sum is equal to

$$2\sum_{s_k \le t_i < s_{k+1}} \mathbb{E}[a_i^2]\Delta t_i^2 \to 0$$

as a_i is bounded.

$\square$

3.1 Ito's formula in multiple dimensions

The same basic idea can be used to prove a multi-dimensional version of Ito's formula, but the calculations are even worse so we omit the proof. We let $B = (B_1, B_2, \ldots, B_m)$ be a m-dimensional Brownian motion generated a filtration $\mathcal{F}_t^{(n)}$ and we let $u = (u_1, \ldots, u_m)$ and (v_{ij}) for $1 \le i \le n$ and $1 \le j \le m$ be so that $u_i \in N(0,t)$ with respect to $\mathcal{F}_t^{(n)}$ for all i and $v = (v_{ij})$ is $\mathcal{F}_t^{(n)}$ for all i, j and

$$\Pr\left[\int_0^t |v_{ij}(s,\omega)| ds < \infty\right] = 1.$$

Then the stochastic integral

$$X = \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s$$

is well-defined. Let $g : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^p$ be C^2. Then if we let $Y(t,\omega) = g(t, X(t,\omega))$, then its kth coordinate Y_k is given by

$$Y_k = \int_0^t \frac{\partial g_k}{\partial t}(s, X_s)ds + \sum_i \int_0^t \frac{\partial g_k}{\partial x_i}(s, X_s)dX_i + \frac{1}{2}\sum_{ij} \int_0^t v_{ij} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(s, X_s)dt$$

where

$$\int f dX_i = \int u_i f dt + \sum_j \int v_{ij} f dB_j.$$

3.2 Applications of Ito's formula

Ito's formula is one of the most important properties of Ito integrals. It sees wide application in financial mathematics, for instance, in the solution of stochastic PDEs such as the Black-Scholes equation, and allows us to calculate Ito integrals that would be difficult otherwise. We demonstrate with an example.

Example 3.1. We calculate

$$\int_0^t B_s dB_s.$$

Let $X_t = B_t = \int_0^t dB_s$ and $g(t,x) = x^2/2$. By Ito's formula, with $u = 1$ and $v = 0$, we have

$$\frac{1}{2}B_t^2 = \int_0^t B_s dB_s + \frac{1}{2}t,$$

so

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t.$$

□

Ito's formula also is a crucial ingredient in the proof of the powerful Martingale Representation Theorem, which we state below without proof.

Theorem 3.2 (Martingale Representation Theorem, [1], p. 80)**.** *Let B be an n-dmensional Brownian motion. Suppose M_t is a $\mathcal{F}_t^{(n)}$ martingale, and $M_t \in L^2(P)$ for all $t \ge 0$. Then there is an unique stochastic process g whose coordinates are in $\mathcal{N}$ so that*

$$M_t = \mathbb{E}[M_0] + \int_0^t g(s,\omega)dB_s;$$

that is, every L^2 martingale can be represented as an Ito integral.

4 Solutions to Stochastic Differential Equations

4.1 Strong Solutions to Stochastic Differential Equations

Let $T > 0$ be fixed and $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, T] \times R^n \to R^{n \times m}$ (there is nothing special about starting at zero, we can start at any $S < T$ and the same results will hold) be sufficiently nice functions (We define exactly what this means below). We seek to find solutions to the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t; \tag{4.1}$$

that is, a process X_t so that

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(t, X_s)dB_s,$$

where B_t is a m-dimensional Brownian motion. We now seek to find suitable conditions on b and σ to ensure that the equation has a unique solution X_t. Consider the two examples below, where no unique solution exists.

Example 4.1. Consider the one-dimensional case. Let $b(t, x) = x^2, \sigma = 0$, and let $X_0 = 0$. Then the equation reduces to a deterministic differential equation $dX_t = b(X_t)dt$. This has unique solution $X(t) = (1 - t)^{-1}$ which in particular is not defined for $t = 1$; hence no global solution exists. In this case, what goes wrong intuitively is that b grows too quickly, and so no solution can exist.

Example 4.2. Again, consider the one-dimensional case. Let $b(t) = 3x^{2/3}, \sigma = 0$ and $X_0 = 0$. Then again the equation is a deterministic differential equation. However, there are infinitely many solutions to this initial value problem; indeed, for all $a > 0$, the function $X(t) = H_a(t)(x - a)^3$ is a solution, where H_a is the shifted Heaviside function which is 0 for $t \leq a$ and 1 for $t > a$. The problem here is that b varies too much at 0; in particular, it is not Lipschitz continuous at zero.

Therefore, to rectify these problems, we specify that b and σ must satisfy that

$$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|) \ \forall t \in [0, T], \forall x \in \mathbb{R}^n \tag{4.2}$$

and

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D|x - y| \ \forall t \in [0, T], \forall x, y \in R^n \tag{4.3}$$

for some $C, D > 0$, where we interpret $|\sigma|$ as the 2-norm of σ interpreted as a $\mathbb{R}^{n \times m}$ vector. Intuitively, condition (4.2) removes the possibility that something like Example 4.1 happens and condition (4.3) removes the possibility that something like Example 4.2 happens.

Theorem 4.1 (Strong Solutions to Stochastic Differential Equations)**.** *Let $\mathcal{F}_t$ be some filtration so that B_t is a Brownian motion with respect to $\mathcal{F}_t$. If b and σ are measurable functions satisfying (4.2) and (4.3), and Z is a measurable function independent of $\mathcal{F}_\infty$, the smallest σ-algebra containing $\mathcal{F}_t$ for all t, then the stochastic differential equation*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \ X_0 = Z \tag{4.4}$$

has an almost surely unique $\mathcal{F}_t$-adapted solution with continuous paths in $\mathcal{N}$.

Proof. We first prove that any solution is unique. Suppose X and X' both solve (4.4). Then

$$\begin{aligned}
\mathbb{E}[|X_t - X'_t|^2] &= \mathbb{E}\left[\left|\int_0^t (b(s,X_s) - b(s,X'_s))ds + \int_0^t (\sigma(s,X_s) - \sigma(s,X'_s))dB_s\right|^2\right] && (4.5)\\
&\leq 2\,\mathbb{E}\left[\left|\int_0^t (b(s,X_s) - b(s,X'_s))ds\right|^2\right] + 2\,\mathbb{E}\left[\left|\int_0^t (\sigma(s,X_s) - \sigma(s,X'_s))dB_s\right|^2\right] && (4.6)\\
&\leq 2t\,\mathbb{E}\left[\int_0^t |b(s,X_s) - b(s,X'_s)|^2 ds\right] + 2\,\mathbb{E}\left[\int_0^t |\sigma(s,X_s) - \sigma(s,X'_s)|^2 ds\right] && (4.7)\\
&\leq 2(t+1)\,\mathbb{E}\left[\int_0^t |b(s,X_s) - b(s,X'_s)|^2 + |\sigma(s,X_s) - \sigma(s,X'_s)|^2 ds\right] && (4.8)\\
&\leq 2D^2(t+1)\,\mathbb{E}\left[\int_0^t |X_s - X'_s|^2 ds\right] = 2D^2(t+1)\int_0^t \mathbb{E}\left[|X_s - X'_s|^2\right] ds && (4.9)
\end{aligned}$$

where the second line follows as $(a+b)^2 \leq 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$, the third line follows from Holder's inequality and Ito's isometry, the fourth line follows trivially from the third, the the final line follows from condition 4.3 and Fubini's theorem. Thus if for all $t \in [0,T]$ we let $v(t) = \int_0^t \mathbb{E}[|X_s - X'_s|^2]ds$ we have that $v(0) = 0$ and $v'(t) \leq 2D^2(T+1)v(t)$; it is readily shown that this implies that $v(t)$ must be constantly zero. Therefore, for each fixed t we have that $\Pr[X_t = X'_t] = 1$, so in particular

$$\Pr[X_t = X'_t,\ \forall t \in \mathbb{Q} \cap [0,T]] = 1$$

as this is a countable union so by the continuity of X_t and X'_t we conclude that

$$\Pr[X_t = X'_t,\ \forall t \in [0,T]] = 1$$

so the solution is unique, as claimed.

We now show that a solution exists. Let $Y_t^0 = Z$ and given Y_t^k define

$$Y_t^{k+1} = Z + \int_0^t b(s, Y_s^k)ds + \int_0^t \sigma(s, Y_s^k)dB_s. \tag{4.10}$$

By assumption, $Y_t^0 \in \mathcal{N}$, and by the same argument as before, we can write that

$$\mathbb{E}[|Y_t^{k+1} - Y_t^k|^2] \leq 2D^2(1+T)\int_0^t \mathbb{E}[|Y_s^k - Y_s^{k-1}|^2]ds$$

for all $k \geq 1, t \in [0,T]$ so in particular, inductively we can show that $Y_t^k \in \mathcal{N}$. As

$$\begin{aligned}
\mathbb{E}[|Y_t^1 - Y_t^0|^2] &= \mathbb{E}\left[\left|\int_0^t b(s,Z)ds + \int_0^t \sigma(s,Z)dB_s\right|^2\right]\\
&\leq 2\,\mathbb{E}\left[\left|\int_0^t b(s,Z)ds\right|^2\right] + 2\,\mathbb{E}\left[\left|\int_0^t \sigma(s,Z)dB_s\right|^2\right]\\
&\leq 2C^2t^2(1 + \mathbb{E}[|Z|]^2) + 2C^2t(1 + \mathbb{E}[|Z|]^2) \leq A_0 t
\end{aligned}$$

where A_0 depends only on C, T and $\mathbb{E}[|Z|]$. Therefore, by repeated integration,

$$\mathbb{E}[|Y_t^{k+1} - Y_t^k|^2] \leq \frac{A_1^{k+1}t^{k+1}}{(k+1)!}$$

where A_1 depends only on C, D, T and $\mathbb{E}[|Z|]$.

Now since

$$\sup_{0\le t\le T}|Y_t^{k+1}-Y_t^k| \le \int_0^T |b(s,Y_s^k)-b(s,Y_s^{k-1})|ds + \sup_{0\le t\le T}\left|\int_0^t (\sigma(s,Y_s^k)-\sigma(s,Y_s^{k-1}))dB_s\right|$$

and if $a \le b+c$ then either $a \le b/2$ or $a \le c/2$. Hence

$$\begin{aligned}\Pr\left[\sup_{0\le t\le T}|Y_t^{k+1}-Y_t^k| > 2^{-k}\right] &\le \Pr\left[\int_0^T |b(s,Y_s^k)-b(s,Y_s^{k-1})|ds > 2^{-k-1}\right]\\ &+ \Pr\left[\sup_{0\le t\le T}\left|\int_0^t (\sigma(s,Y_s^k)-\sigma(s,Y_s^{k-1}))dB_s\right| > 2^{-k-1}\right].\end{aligned}$$

By Doob's martingale inequality, we have that

$$\begin{aligned}\Pr\left[\sup_{0\le t\le T}\left|\int_0^t (\sigma(s,Y_s^k)-\sigma(s,Y_s^{k-1}))dB_s\right| > 2^{-k-1}\right] &\le 2^{2k+2}E\left[\left|\int_0^t (\sigma(s,Y_s^k)-\sigma(s,Y_s^{k-1}))dB_s\right|^2\right]\\ &= 2^{2k+2}E\left[\int_0^t |(\sigma(s,Y_s^k)-\sigma(s,Y_s^{k-1}))|^2 ds\right]\\ &= 2^{2k+2}\int_0^t \mathbb{E}[|(\sigma(s,Y_s^k)-\sigma(s,Y_s^{k-1}))|^2]ds\end{aligned}$$

by Ito's isometry and Fubini's theorem.

Hence as by Markov's inequality we have

$$\begin{aligned}\Pr\left[\int_0^T |b(s,Y_s^k)-b(s,Y_s^{k-1})|ds > 2^{-k-1}\right] &\le 2^{2k+2}\left[\int_0^t |b(s,Y_s^k)-b(s,Y_s^{k-1})|ds\right]^2\\ &\le 2^{2k+2}\int_0^T |b(s,Y_s^k)-b(s,Y_s^{k-1})|^2 ds\end{aligned}$$

where the second line follows by the Cauchy-Schwarz inequality for integrals. Therefore

$$\begin{aligned}\Pr\left[\sup_{0\le t\le T}|Y_t^{k+1}-Y_t^k| > 2^{-k}\right] &\le 2^{2k+2}\left(\int_0^T |b(s,Y_s^k)-b(s,Y_s^{k-1})|^2 + D[|(\sigma(s,Y_s^k)-\sigma(s,Y_s^{k-1}))|^2]ds\right)\\ &\le 2^{2k+2}D^2(T+1)\int_0^t \frac{A_1^k t^k}{k!}dt.\end{aligned}$$

Thus, if we choose $A_1 \ge 4D^2(T+1)$ we have that

$$\Pr\left[\sup_{0\le t\le T}|Y_t^{k+1}-Y_t^k| > 2^{-k}\right] \le \frac{(4A_1T)^{k+1}}{k+1}.$$

As the right hand side is summable, by Borel-Cantelli we have that

$$\Pr\left[\sup_{0\le t\le T}|Y_t^{k+1}-Y_t^k| > 2^{-k} \text{ i. o.}\right] = 0.$$

Therefore for almost all ω and the sequence $Y_t^k(\omega)$ is uniformly convergent for $t \in [0,t]$; thus an almost sure limit X_t exists. As Y_t^k are t-continuous for almost all ω, so is X_t. We claim that $Y_t^k \to X_t$ in L^2 (so $X_t \in N$) and that X_t solves the differential equation.

But we previously showed that

$$\|Y_t^{k+1} - Y_t^k\|_{L^2} \leq \left(\frac{(A_2 T)^{k+1}}{(k+1)!} \right)^{1/2}$$

so in particular for $m \geq n$ we have

$$\|Y_t^m - Y_t^n\|_{L^2} \leq \sum_{k=n}^{m-1} \left(\frac{(A_2 T)^{k+1}}{(k+1)!} \right)^{1/2} \to 0$$

as $m, n \to \infty$ so a L^2 limit Y_t of the sequence exists; thus a subsequence converges pointwise to Y_t so $Y_t = X_t$ so Y_t^k converges in L^2 to X_t.

Finally, we seek to show that X_t solves the differential equation, as claimed. It suffices to show that

$$\int_0^t \sigma(s, Y_s^k) dB_s \to \int_0^t \sigma(s, X_s) dB_s$$

and

$$\int_0^t b(s, Y_s^k) ds \to \int_0^t b(s, X_s) ds$$

in L^2.

But since by Ito's isometry and Fubini's theorem

$$\begin{aligned} \mathbb{E}\left[\left| \int_0^t \sigma(s, Y_s^k) dB_s - \sigma(s, X_s) dB_s \right|^2 \right] &= \int_0^t \mathbb{E}\left[|\sigma(s, Y_s^k) - \sigma(s, X_s)|^2 \right] ds \\ &\leq D^2 \int_0^t \|Y_s^k - X_s\|_{L^2} ds \to 0 \end{aligned}$$

and by Holder's inequality

$$\begin{aligned} \mathbb{E}\left[\left| \int_0^t b(s, Y_s^k) - b(s, X_s) ds \right|^2 \right] &\leq t\, \mathbb{E}\left[\int_0^t |b(s, Y_s^k) - b(s, X_s)|^2 ds \right] \\ &\leq TD^2\, \mathbb{E}\left[\int_0^t \|Y_s^k - X_s\|_{L^2} \right] ds \to 0 \end{aligned}$$

so we are done. □

4.2 Weak Solutions to Stochastic Differential Equations

A slightly weaker notion of uniqueness of solutions also exists.

Definition 4.1. A *weak solution* (X, B, P) to the SDE is a pair of processes (X_t, B_t) and a probability measure P so that B is a Brownian motion under P and (4.4) holds under P for X. Solutions to (4.4) are *weakly unique* if for all weak solutions (X, B, P) and (X', B', P'), the joint law of (X, B) under P is the same as the joint law of (X', B') under P'.

It turns out that under very general conditions, weak uniqueness holds, and in particular, if pathwise uniqueness holds, then weak uniqueness holds, under very general conditions on b and σ. See [6] for details. Under our assumptions, the proof that weak uniqueness holds is not difficult.

Theorem 4.2. *Suppose b and σ satisfy conditions 4.2 and 4.3. Then weak uniqueness holds for (4.4), if $Z = x_0$ is deterministic.*

Proof. Let (X, B, P) and $(\tilde{X}, \tilde{B}, \tilde{P})$ be two weak solutions to (4.4). Let $Y_t^0 = x_0$ and define Y_t^k as in (4.10) using probability law P and similarly define $\tilde{Y}_t^0 = x_0$ and $\tilde{Y}_t^k$ as in (4.10) using $\tilde{P}$. Since B is a Brownian motion under P and $\tilde{B}$ is a Brownian motion under $\tilde{P}$, (Y^0, B) and $(\tilde{Y}^0, B')$ have the same law under P and $\tilde{P}$, respectively; inductively, it is not hard to see that this implies that (Y^k, B) and $(\tilde{Y}^k, B')$ have the same laws under P and $\tilde{P}$ respectively, so as $Y^k \to X$ almost surely and in $L^2(P)$ and $\tilde{Y}^k \to \tilde{X}$ almost surely and in $L^2(\tilde{P})$ the laws of (X, B) and $(\tilde{X}, \tilde{B})$ must be the same as well. $\square$

5 Ito Diffusions

5.1 Notation

Fix some b, σ functions satisfying conditions 4.2 and 4.3. We also assume that b and σ only depend on x; that is, there is no time dependence. Denote by $X_s^{t,x}$ the almost surely unique solution to the equation

$$dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dB_s, s \geq t;\ X_s^{t,x} = x \tag{5.1}$$

where x is some scalar constant. When $t = 0$ let $X_s^x = X_s^{0,x}$. The resulting process has the property of being time-homogenous. We call such a process an *Ito diffusion*.

5.2 The Markov Property for Ito Diffusions

We first need a lemma.

Lemma 5.1. *For all $s \geq t$, the random variable $X_t^{s,x}$ is independent of $\mathcal{F}_t$.*

Proof. Recall the proof of existence: we let $X_s^0 = x$ and inductively let

$$X_s^{k+1} = X_0 + \int_0^t b(X_s^k)ds + \int_0^t \sigma(Y_s^k)dB_s$$

and we let X_t be the almost sure and L^2 limit of the X_t^k. But clearly X_s^0 is independent of $\mathcal{F}_t$, and inductively, by the independence of increments of Brownian motion, if X_s^k is independent of $\mathcal{F}_t$ so is X_s^{k+1}, so independence holds for all k, and in particular, holds when we pass to the limit, so $X_s^{x,t}$ is independent of $\mathcal{F}_t$. $\square$

Theorem 5.2 (The Strong Markov Property for Ito Diffusions)**.** *Let f be a bounded Borel function from $\mathbb{R}^n \to \mathbb{R}$. Then for τ an almost surely finite stopping time, $h \geq 0$*

$$\mathbb{E}[f(X_{\tau+h}^x)|\mathcal{F}_\tau](\omega) = \mathbb{E}[f(X_h^y)]|_{y=X_\tau^x(\omega)}. \tag{5.2}$$

Remark 5.1. All the notation on right side means is that if we let $h(y) = \mathbb{E}[f(X_h^y)]$ then $\mathbb{E}[f(X_{\tau+h}^x)|\mathcal{F}_t](\omega) = h(X_\tau^x(\omega))$; notice that in general $h(X_\tau^x(\omega)) \neq \mathbb{E}[f(X_h^{X_\tau(\omega)})]$.

Proof. By linearity of Ito's integral, we have that

$$X_s^x = X_\tau^x + \int_\tau^{\tau+h} b(X_u)du + \int_\tau^{\tau+h} \sigma(X_u)dB_u.$$

By the strong Markov property for Brownian motion, the process $\tilde{B}_h = B_{\tau+h} - B_\tau$ is a Brownian motion with respect to the filtration $\{\mathcal{F}_{\tau+h}\}_{h\geq 0}$ independent of $\mathcal{F}_\tau$, and from there it is not hard to see that

$$X_\tau^x + \int_0^h b(X_{\tau+u})du + \int_0^h \sigma(X_{\tau+u})d\tilde{B}_u$$

is well defined and equal to the above. Hence if we let $F(x,t,s,\omega)$ to be the (unique) solution $\tilde{X}_s^{x,t}(\omega)$ of

$$\tilde{X}_s^{x,t} = x + \int_t^s b(\tilde{X}_u^{x,t})du + \int_t^s \sigma(\tilde{X}_u^{x,t})d\tilde{B}_u$$

by strong uniqueness $X_s^x(\omega) = X_s^{t,X_t^x}(\omega)$ [1] so

$$X_{\tau+h}(\omega) = F(X_t^x, 0, h, \omega)$$

for all $h \geq 0$. In our new notation, Equation 5.2 becomes equivalent to showing that

$$\mathbb{E}[f(F(X_\tau^x(\omega), 0, h, \omega))|\mathcal{F}_\tau] = \mathbb{E}[f(F(x,0,h,\omega))]|_{x=X_\tau(\omega)} \tag{5.3}$$

Let $g(x,\omega) = f(F(x,0,h,\omega))$. It is left to the reader to check that $g(x,\omega)$ is in fact measurable (see [5] exercise 7.6). Approximate g from below by a sequence of functions of the form

$$g_n(x) = \sum_{k=1}^m \phi_{nk}(x)\psi_{nk}(\omega).$$

Then we have that

$$\begin{aligned}
\mathbb{E}[g(X_t^x(\omega),\omega)|\mathcal{F}_t] &= \lim_{n\to\infty} \mathbb{E}[g_n(X_\tau^x(\omega),\omega)|\mathcal{F}_\tau] && (5.4)\\
&= \lim_{n\to\infty} \sum_{k=1}^m \phi_{nk}(X_\tau^x(\omega))\, \mathbb{E}[\psi_{nk}(\omega)|\mathcal{F}_\tau] && (5.5)\\
&= \lim_{n\to\infty} \sum_{k=1}^m \mathbb{E}[\phi_{nk}(y)\psi_{nk}(\omega)|\mathcal{F}_\tau]|_{y=X_\tau^x(\omega)} && (5.6)\\
&= \mathbb{E}[g(y,\omega)|\mathcal{F}_\tau]|_{y=X_\tau(\omega)} = \mathbb{E}[g(y,\omega)]|_{y=X_\tau(\omega)} && (5.7)
\end{aligned}$$

Step (5.4) follows from the dominated convergence theorem applied to $g(X_t^x(\omega),\omega)$. Step (5.5) follows from the measurability of X_τ^x with respect to $\mathcal{F}_\tau$. Step (5.6) follows because $\phi_{nk}(y)$ is simply a constant in that expectation, so we can always move it in and out, without trouble. Finally, step (5.7) follows from Lemma 5.1 applied to the differential equation with respect to $\tilde{B}$.

□

In particular, if we let $\tau = t$, we get that Ito diffusions satisfy the standard Markov property.

Corollary 5.3 (The weak Markov Property for Ito Diffusions)**.** *Let f be a bounded Borel function from $\mathbb{R}^n \to \mathbb{R}$. Then for $t,h \geq 0$*

$$\mathbb{E}[f(X_{t+h}^x)|\mathcal{F}_t](\omega) = \mathbb{E}[f(X_h^y)]|_{y=X_t^x(\omega)}. \tag{5.8}$$

We now seek to prove another version of the Strong Markov Property which will become very important for the solution of the stochastic Dirichlet problem. Suppose for now that $\Omega = C([0,\infty))$; we mentioned previously that we can always assume that this is the case. For $t > 0$ define the shift operator $\theta_t : \Omega \to \Omega$ by simply shifting; that is, if $\omega(s)$ is a continuous function, then $\theta_t(\omega)(s) = \omega(s+t)$.

For $x \in \mathbb{R}^n$, let $\mathcal{M}_x$ be the σ-algebra generated by the random variables $X_t^x(\omega)$ for $t \geq 0$. Let $\mathcal{J}_x$ be the space of $\mathcal{M}_x$ measurable functions. The following is an elementary measure theoretic property.

Proposition 5.4. *All functions $\eta \in \mathcal{J}_x$ can be represented as increasing, almost sure limits of sums of functions of the form $\prod_i g_i(X_{t_i}^x)$ where g_i are Borel measurable functions,.*

[1] Notice that technically here the initial condition does not satisfy the assumptions in the theorem stated previously; however the proof of strong uniqueness did not depend on it

Proof. This follows immediately from the monotone class theorem as finite dimensional cylinders are π-system which generate $\mathcal{M}$ and all characteristic functions of cylinders can be represented in the form above. $\square$

For each $x, y \in \mathbb{R}^n$, we define the probability measure $P^{x,y}$ on $\mathcal{M}_x$ to be the measure given by the law of X^y; namely, $P^y[X_t^x \in A] = P[X_t^y \in A]$ for all A Borel, where P is the original probability measure. It can be shown that this indeed defines a measure (the law of X^x) ([4], p 22), and that

$$\mathbb{E}^y[g_1(X_{t_1}^x)\dots g_k(X_{t_k}^x)] = \mathbb{E}[g_1(X_{t_1}^y)\dots g_k(X_{t_k}^y)]$$

where $E^{x,y}$ is expectation with respect to $P^{x,y}$ ([5], p. 87).

Theorem 5.5 (The Strong Markov Property for Ito Diffusions (Ver. 2))**.** *Let $\eta \in \mathcal{J}_x$ and τ be any stopping time which is almost surely finite. Then*

$$\mathbb{E}[\eta \circ \theta_\tau | \mathcal{F}_\tau] = \mathbb{E}^{x,y}[\eta]|_{y=X_\tau} \tag{5.9}$$

Proof. It suffices to check that this condition holds whenever $\eta = g_1(X_{t_1}^x)\dots g_k(X_{t_k}^x)$. In that case the above reduces to showing that

$$\mathbb{E}[g_1(X_{t_1+\tau}^x)\dots g_k(X_{t_k+\tau}^x)|\mathcal{F}_\tau] = \mathbb{E}[g_1(X_{t_1}^y)\dots g_k(X_{t_k}^y)]|_{y=X_\tau}.$$

Without loss of generality assume that $t_1 \leq t_2 \leq \ldots \leq t_k$. We proceed by induction on k, the number of elements in the product. This holds by the previous version of the strong Markov property when $k = 1$. Suppose inductively this holds for all $k_0 \leq k$.

$$\begin{aligned}
\mathbb{E}[g_1(X_{t_1+\tau}^x)\dots g_k(X_{t_k+\tau}^x)|\mathcal{F}_\tau] &= \mathbb{E}[\mathbb{E}[g_1(X_{t_1+\tau}^x)\dots g_{k-1}(X_{t_{k-1}+\tau}^x) g_k(X_{t_k+\tau}^x)|\mathcal{F}_{\tau+t_{k-1}}]|\mathcal{F}_\tau] \\
&= \mathbb{E}[g_1(X_{t_1+\tau}^x)\dots g_{k-1}(X_{t_{k-1}+\tau}^x)\,\mathbb{E}[g_k(X_{t_k+\tau}^x)|\mathcal{F}_{\tau+t_{k-1}}]|\mathcal{F}_\tau] \\
&= \mathbb{E}[g_1(X_{t_1+\tau}^x)\dots g_{k-1}(X_{t_{k-1}+\tau}^x)(\mathbb{E}[g_k(X_{t_k-t_{k-1}}^y)]|_{y=X^{x_k}_{\tau+t_{k-1}}})|\,\mathcal{F}_\tau] \\
&= \mathbb{E}[g_1(X_{t_1}^z)\dots g_{k-1}(X_{t_{k-1}}^z)\,\mathbb{E}[g_k(X_{t_k-t_{k-1}}^y)]|_{y=X_{t_{k-1}}^z}]|_{z=X_\tau} \\
&= \mathbb{E}[g_1(X_{t_1}^z)\dots g_k(X_{t_k}^z)]_{z=X_\tau}
\end{aligned}$$

where the second equality follows from the tower property of conditional expectation and since $g_i(X_{t_i+\tau})$ is $\mathcal{F}_{\tau+t_{k-1}}$ measurable for $i \leq k-1$, the third line follows from the previous version of the strong Markov property, the fourth line follows from applying the inductive hypothesis to the function

$$g_1(X_{t_1+\tau}\dots g_{k-2}(X_{t_{k-2}+\tau})\tilde{g}_{k-1}(X_{t_{k-1}+\tau})$$

where $\tilde{g}_{k-1}(x) = g_{k-1}(x)E^x[g_k(X_{t_k-t_{k-1}})]$ and the last line follows from the weak Markov property. $\square$

6 Generalizations of the Dirichlet Problem

In this thesis, we will primarily consider solutions to the *stochastic Dirichlet problem* which we state below, and ostensibly weaker equation than the *deterministic generalized Dirichlet problem*; however, it will turn out that under a very general set of conditions, the two are in fact equivalent.

6.1 The Stochastic Dirichlet Problem

We again tacitly assume that $\Omega = C([0,\infty))$. Let b and σ satisfy conditions 4.2 and 4.3, along with the usual measurability conditions, so that for all $x \in \mathbb{R}^n$,

$$dX_t = b(X_s)ds + \sigma(X_s)dB_s,\ X_0 = x$$

has a unique solution, which we denote X_t^x. Let $D \subset \mathbb{R}^n$ be a domain. Let $\tau_D(\omega) = \inf\{t : X_t^x(\omega) \notin D\}$ be the exit time for D and suppose that $\tau_D < \infty$ except possibly on a set with probability zero for all $x \in D$. [2] We now define the stochastic version of harmonicity using a stochastic notion of the mean value property.

[2]there is a tacit dependence on x in the definition of τ, but it will not matter to us, so we omit it, and it should always be clear from context what x is.

Definition 6.1. Let f be a locally bounded, measurable function on D. Then f is X-harmonic in D if $f(x) = E(f(X^x_{\tau_U}))$ for all U bounded and open with $\overline{U} \subset D$.

Thus, the formulation of the stochastic Dirichlet problem is the following: given a bounded, Borel measurable function ϕ on ∂D, find a function $\tilde{\phi}$ on D so that

1. $\tilde{\phi}$ is X-harmonic.
2. For all $x \in D$,

$$\lim_{t \to \tau_D} \tilde{\phi}(X^x_t) = \phi(X^x_{\tau_D})$$

almost surely.

Theorem 6.1. *Let ϕ be a bounded, measurable function on ∂D. Define*

$$\tilde{\phi}(x) = \mathbb{E}[\phi(X^x_{\tau_D})].$$

Then $\tilde{\phi}$ is the unique bounded X-harmonic function so that condition (2) above holds.

Proof. We first show that $\tilde{\phi}$ is X-harmonic, so we need to show that

$$\tilde{\phi}(x) = \mathbb{E}[\tilde{\phi}(X^x_{\tau_U})]$$

for all U bounded and open with $\overline{U} \subset D$. Fix such a U. Clearly as $\tau_D < \infty$ almost surely so is $\tau_U = \inf\{t : X^x_t \notin U\}$. Extend ϕ to be zero outside ∂D, and let ϕ_n be a sequence of bounded continuous functions so that $\phi_n \to \phi$ almost everywhere. Let $\eta_n = \phi_n(X^x_{\tau_D})$. It is not hard to show that ϕ_n and ϕ are in $\mathcal{J}_x$. We claim that almost surely,

$$\eta_n \circ \theta_{\tau_U} = \eta_n$$

simply because $\tau_U \leq \tau_D$, since we must hit U before we hit D. Therefore, by the Strong Markov property, version 2, we get that

$$\mathbb{E}[\phi_n(X^x_{\tau_D}) | \mathcal{F}_t] = \mathbb{E}^{x,y}[\eta_n]|_{y = X_{\tau_U}}$$

It is not hard to check that $\mathbb{E}^{x,y}[\phi_n(X^x_{\tau_D})] = \mathbb{E}[\phi_n(X^y_{\tau_D})]$ (simply approximate τ_D pointwise) so by taking expections we get that ϕ_n is X-harmonic. As $\phi_n(X^x_{\tau_D}) \to \phi(X^x_{\tau_D})$ almost surely and the functions are bounded, by bounded convergence we get that ϕ is also X-harmonic.

We now show that condition (2) above holds. Fix $x \in D$. Let $\{D_k\}$ be an increasing sequence of open sets such that $\overline{D_k} \subset D$ and $D = \cup D_k$. Let $\tau_k = \tau_{D_k}$. Then by the strong Markov property (version 2) with $\eta = \phi(X_{\tau_D})$

$$\tilde{\phi}(X^x_{\tau_k}) = \mathbb{E}[\phi(X^y_{\tau_D})]|_{y = X^x_{\tau_k}} = \mathbb{E}[\theta_{\tau_k} \phi(X^x_\tau) | \mathcal{F}_{\tau_k}] = \mathbb{E}[\phi(X^x_\tau) | \mathcal{F}_{\tau_k}]$$

where the last equality follows from what we argued before. But then $M_k = \mathbb{E}[\phi(X^x_\tau) | \mathcal{F}_{\tau_k}]$ is a bounded (as ϕ is bounded) discrete time martingale, so by the martingale convergence theorem $\tilde{\phi}(X_{\tau_k}) \to \phi(X_\tau)$ almost surely and in L^p for all $p < \infty$.

Thus we have shown that for a particular sequence of times, the desired limit holds. We now wish to show that this limit holds for all times t. But since for all k and for all t the function $\gamma_t = \tau_k \vee (t \wedge \tau_{k+1})$ is a stopping time, by the same computation as we did above the process

$$N_t = \tilde{\phi}(X^x_{\gamma_t}) - \tilde{\phi}(X^x_{\gamma_k})$$

is a martingale with respect to $\{\mathcal{F}_{\gamma_t}\}_{t \geq 0}$, so by Doob's martingale inequality, for all $\epsilon > 0$,

$$P\left[\sup_{\tau_k \leq t \leq \tau_{k+1}} |\tilde{\phi}(X^x_t) - \tilde{\phi}(X^x_{\gamma_k})| > \epsilon\right] \leq \frac{1}{\epsilon^2} \mathbb{E}[|\tilde{\phi}(X^x_t) - \tilde{\phi}(X^x_{\gamma_k})|^2] \to 0$$

as $k \to \infty$, so this combined with the fact that $\tilde{\phi}(X_{\tau_k}) \to \phi(X_\tau)$ almost surely immediately gives us (2).

Finally, we demonstrate uniqueness of the solution. Suppose g is bounded, X-harmonic, and let D_k and τ_k be as above. Then

$$g(x) = \lim_{k\to\infty} \mathbb{E}(g(X^x_{\tau_k})) = \mathbb{E}[\phi(X^x_{\tau_D})] = \tilde{\phi}(x)$$

where the first inequality follows from X-harmonicity and the second follows from (2) and bounded convergence. Hence we are done. $\square$

6.2 The Characteristic Operator

The following discussion follows [5]. Let L be a semi-elliptic partial differential operator on $C^2(\mathbb{R}^n)$ of the form

$$L = \sum_{ij} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x)\frac{\partial}{\partial x_i}.$$

Let $D \subset \mathbb{R}^n$ be a domain, and let ϕ be a continuous function defined on ∂D. Our goal ultimately is to find a function $\tilde{\phi}$ defined on D so that $L\tilde{\phi} = 0$ and $\lim_{x\to x_0} \tilde{\phi}(x) = \phi(x_0)$ for all "reasonable" $x_0 \in \partial D$. We call $\tilde{\phi}$ the L-harmonic extension of ϕ to D.

We seek to connect the solution to our stochastic Dirichlet problem to the solution of this deterministic generalized Dirichlet problem.

Proposition 6.2. *Let Y^x_t be a $\mathbb{R}^n$ dimensional stochastic integral of the form*

$$Y^x_t(\omega) = x + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s(\omega)$$

where u has coordinates in $\mathcal{N}$ and v satisfies (2.2). Let $\mathbb{E}^x$ be expectation with respect to the law of Y^x_t. Let $f \in C^2(\mathbb{R}^n)$ and let τ be a stopping time with respect to $\mathcal{F}_t$ with $\mathbb{E}^x[\tau] < \infty$. Assume u and v are bounded on the set where $Y(t,\omega) \in supp f$. Then

$$\mathbb{E}^x[f(Y_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau \left(\sum_i u_i(s,\omega)\frac{\partial f}{\partial x_i}(Y_s) + \frac{1}{2}\sum_{i,j}(vv^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}(Y_s)\right) ds\right].$$

Proof sketch. We will proceed a little bit informally; the details are tedious, boring and ultimately unenlightening. Let $Y = (Y_1, \ldots, Y_n)$ and $B = (B_1, \ldots, B_m)$. We let $Z = f(Y)$, then apply the differential form of the multi-dimensional Ito's formula:

$$\begin{aligned} dZ &= \sum_i \frac{\partial f}{\partial x_i}(Y)dY_i + \frac{1}{2}\sum_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}(Y)dY_i dY_j \\ &= \sum_i u_i \frac{\partial f}{\partial x_i}dt + \frac{1}{2}\sum_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}(vdB)_i(vdB)_j + \sum_i \frac{\partial f}{\partial x_i}(vdB)_i. \end{aligned}$$

By the rules for the (informal) differential calculus it turns out that

$$(vdB)_i(vdB)_j = \sum_k v_{ik}v_{jk}dt = (vv^T)_{ij}dt.$$

Thus

$$dZ = \left(\sum_i u_i\frac{\partial f}{\partial x_i} + \frac{1}{2}(vv^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}\right)dt + \sum_{i,k} v_{ik}\frac{\partial f}{\partial x_i}dB_k.$$

Through formal arguments using the finiteness of the quadratic variation of the Brownian motion as we did in the proof of Ito's formula, the above is equivalent to the following equality:

$$Z_t = f(Y_t) = \int_0^t \sum_i u_i(s,\omega)\frac{\partial f}{\partial x_i}(Y_s) + \frac{1}{2}(vv^T)_{ij}(s,\omega)\frac{\partial^2 f}{\partial x_i \partial x_j}(Y_s)ds + \sum_{ik}\int_0^t v_{ik}\frac{\partial f}{\partial x_i}(Y_s)d(B_k)_s. \qquad (6.1)$$

Moreover, let g be a Borel function with $|g| \leq M$ for some $M > 0$. Then for all integers k we have

$$\mathbb{E}^x\left[\left(\int_0^{\tau\wedge k} g(Y_s)dB_s\right)\right] = \mathbb{E}^x\left[\left(\int_0^k \chi_{s<T}\, g(Y_s)dB_s\right)\right] = 0$$

by Proposition 2.4, as the integrand $\chi_{s<T}\, g(Y_s) \in \mathcal{N}(0,T)$. Furthermore, since

$$\mathbb{E}^x\left[\left(\int_0^{\tau\wedge k} g(Y_s)dB_s\right)^2\right] = \mathbb{E}^x\left[\int_0^{\tau\wedge k} g^2(Y_s)ds\right] \leq M^2\,\mathbb{E}^x[\tau] < \infty$$

so the family $\{\int_0^{\tau\wedge k} g(Y_s)dB_s\}_k$ is uniformly integrable, so by [5] Appendix C we may push the limit into the integral and we obtain that

$$0 = \lim_{k\to\infty}\mathbb{E}^x\left[\int_0^{\tau\wedge k} g(Y_s)dB_s\right] = \mathbb{E}^x\left[\lim_{k\to\infty}\int_0^{\tau\wedge k} g(Y_s)dB_s\right] = \mathbb{E}^x\left[\int_0^{\tau} g(Y_s)dB_s\right]$$

so by taking expectations in (6.1) we are done. □

The relationships between the stochastic differential equation and the generalized Dirichlet equation become apparent through an operator called the *characteristic operator*.

Definition 6.2. Let $\{X_t^x\}_x$ be the solutions to a time-homogeneous Ito diffusion in $\mathbb{R}^n$ starting at x, where $x \in \mathbb{R}^n$. The characteristic operator A of X_t is given by

$$Af(x) = \lim_{t\downarrow 0}\frac{\mathbb{E}[f(X_t^x)] - f(x)}{t}.$$

The set of functions f for which this characteristic operator exists is denoted by $\mathcal{D}_A$.

We prove some important properties of the characteristic operator below.

Theorem 6.3. *Let X_t be an Ito diffusion*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t$$

where b and σ satisfy (4.2) and (4.3). If A is the characteristic operator for X_t and $f \in C^2(\mathbb{R}^n)$ has bounded first and second derivatives then $f \in \mathcal{D}_A$ and

$$Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{ij}(\sigma\sigma^T)_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Proof. By Proposition 6.2 with $\tau = t$ we have

$$f(X_t^x) = f(x) + \mathbb{E}^x\left[\int_0^t\left(\sum_i b_i(s)\frac{\partial f}{\partial x_i}(Y_s) + \frac{1}{2}\sum_{ij}(\sigma\sigma^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}(Y_s)\right)ds\right]$$

so by the fundamental theorem of calculus the desired result follows immediately. □

Plugging this into the above, we get the following:

Corollary 6.4 (Dynkin's Formula). *Let $f \in C^2(\mathbb{R}^n)$. Let τ be a stopping time with $\mathbb{E}^x[\tau] < \infty$. Then*

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau Af(X_s)ds\right].$$

The final important ingredient is the following theorem, which is too involved for us to prove here.

Theorem 6.5 ([5], p. 96). *Let X_t be an Ito diffusion, and A is the characteristic operator of X. Then if $f \in C^2(\mathbb{R}^n)$, and U_k is a sequence of open sets $\cap_k U_k = \{x\}$ we have*

$$Af(x) = \lim_{k\to\infty} \frac{E^x[f(X_{\tau_{U_k}})] - f(x)}{\mathbb{E}^x[\tau_{U_k}]}$$

where τ_{U_k} is the hitting time for U_k.

6.3 Solutions to the Deterministic Generalized Dirichlet Problem

We are now prepared to give an exposition on solutions of the deterministic generalized Dirichlet problem. Let b be the vector of b_i. Suppose we can find σ so that $\frac{1}{2}\sigma\sigma^T = a$ where a is the matrix of a_{ij}s, and that b and σ satisfy conditions 4.2 and 4.3. It turns out that as long as we impose some boundedness conditions on a_{ij} then such a σ satisfying the desired conditions can always be found.

Moreover, by Theorem 6.3 and Theorem 6.5, if X_t is an Ito diffusion b and σ then if ϕ is X-harmonic and $\phi \in C^2(\mathbb{R}^n)$ with bounded first and second derivatives then ϕ is L-harmonic as well. One could therefore conjecture that the solution of the stochastic Dirichlet problem, if it exists, is the solution to to the generalized Dirichlet Problem. This is unfortunately not true in general, because the boundary behavior can be strange. It turns out, however, that if the domain satisfies satisfies *Hunt's condition* ([5], p. 140), then the L-harmonic extension of ϕ, if it exists, will be equal to the solution to the stochastic Dirichlet problem, as defined above. The question of existence of solutions to deterministic generalized Dirichlet problems is however too involved for us here.

References

[1] Richard F. Bass. *Stochastic Processes*. Cambridge, 2011.

[2] Rick Durrett. *Probability: Theory and Examples*. Cambridge, 2010.

[3] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*. Springer, 2 edition, 1991.

[4] Thomas Liggett. *Stochastic Differential Equations: An Introduction with Applications*. Springer-Verlag, 2010.

[5] Bernt Øksendal. *Continuous Time Markov Processes: An introduction*. American Mathematical Society, Berlin, 3rd edition, 1980.

[6] D. Stroock and S.R.S. Varadhan. *Multidimensional Diffusion Processes*. Springer-Verlag, 1979.